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WHEN IS A LOTS DENSELY ORDERABLE?

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We obtain sufficient conditions for a linearly ordered topological space (LOTS) to be densely orderable. As a consequence we obtain a characterization of those metrizable LOTS with connected ordered compactifications.

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1. Introduction

A well-known result states that the weight w of a linearly ordered topological space (LOTS) is equal to the sum of the density d and the number of jumps j. Thus if $j > d$, then $w = j$ and hence j is an order-topological invariant of the space in the sense that any other order which induces the same topology will have the same number of jumps. On the other hand, if $j \leqslant d$ (in particular if $j = \aleph_0$) then this is not necessarily the case—for example, the space of rationals Q with the order topology is homeomophic to $Q \times \{0, 1\}$ with the lexicographic order topology. In this paper we will obtain conditions under which a LOTS can be reordered without jumps, that is to say under which there exists a dense order inducing the same topology.

2. Some combinatorial preliminaries

Recall that a digraph is a pair (V, A) where V is a set and A is a binary relation on V. If $D = (V, A)$ is a digraph and π is a partition of V, we write $u \equiv w (\mathrm{mod}\ \pi)$ to mean that $u, w \in V$ lie in the same equivalence class of π. We further define

$$A(\pi, 0) = \{(u, w) \in A\colon u \equiv w (\mathrm{mod}\ \pi)\}, \quad A(\pi, 1) = A - A(\pi, 0),$$

$D(\pi, 0) = (V, A(\pi, 0))$ and $D(\pi, 1) = (V, A(\pi, 1))$.

Let $(I, <)$ be a directed set, $D = (V, A)$ a digraph and $\{D_i = (V, A_i): i \in I\}$ a family of digraphs. We will say that D_i converges to D (and write Lim $D_i = D$ or $D_i \to D$) if for each $u, w \in V$, there exists $i(u, w) \in I$ such that

(a) $(u, w) \in A$ and $j \geqslant i(u, w)$ then $(u, w) \in A_j$, or

(b) $(u, w) \notin A$ and $j \geqslant i(u, w)$ then $(u, w) \notin A_j$.

If $D_i \to D$ then we say that D is the limit of the net $\{D_i : i \in I\}$, and we note that the limit is unique. We omit the easy proofs of Lemmas 2.1, 2.2 and Theorem 2.3.

2.1. Lemma. *A* net $\{D_i : i \in I\}$ *of digraphs is convergent if and only if for each* $u, w \in V$ *there exists* $i(u, w) \in I$ *such that*

(a) $j \geqslant i(u, w)$ *implies that* $(u, w) \in A_j$, *or*

(b) $j \geqslant i(u, w)$ *implies that* $(u, w) \notin A_j$.

2.2. Lemma. *Let* $\{D_i = (V, A_i): i \in I\}$ *be a net of digraphs and* π *any partition of* V. *Then* $\{D_i: i \in I\}$ *is convergent if and only if* $\{D_i(\pi, 0): i \in I\}$ *and* $\{D_i(\pi, 1): i \in I\}$ *are convergent; moreover,* $\lim D_i = \lim D_i(\pi, 0) \cup \lim D_i(\pi, 1)$.

2.3. Theorem. *Suppose that* $D_i \to D$ *then: If* D_i *is a poset (respectively, a linearly ordered set) for each* $i \geqslant i_0$, *then* D *is a poset (respectively, linearly ordered set); moreover, if* S *is a convex subset of* D_i *for each* $i \geqslant i_0$, *then* S *is convex in* D.

2.4. Theorem. *Let* $(I, <)$ *be a directed set,* $\{D_i = (V, A_i): i \in I\}$ *a net of digraphs,* $\{\pi_i : i \in I\}$ *a monotone family of partitions of* V *(that is a family of partitions with the property that* π_j *refines* π_i *whenever* $j \geqslant i$*) and* π *the smallest common refinement of the* π_i. *If*

(a) $j > i$ *implies that* $D_j(\pi_i, 1) = D_i(\pi_i, 1)$ *and*

(b) $\{D_i(\pi, 0): i \in I\}$ *is convergent,*

then $\{D_i : i \in I\}$ *is convergent.*

Proof. Because of condition (b) and Lemma 2.2 we need only show that $\{D_i(\pi, 1): i \in I\}$ is convergent. If $u \equiv w (\text{mod } \pi)$, then for each j we have $(u, w) \notin A_j(\pi, 1)$. On the other hand if $u \not\equiv w (\text{mod } \pi)$ then there exists $i(u, w)$ such that $u \not\equiv w (\text{mod } \pi_{i(u,w)})$. If $j \geqslant i(u, w)$ then clearly $u \not\equiv w (\text{mod } \pi_j)$. Condition (a) implies that $(u, w) \in A_j(\pi_{i(u,w)}, 1)$ if and only if $(u, w) \in A_{i(u,w)}(\pi_{i(u,w)}, 1)$ and so $(u, w) \in A_j(\pi, 1)$ for each $j \geqslant i(u, w)$ if and only if $(u, w) \in A_{i(u,w)}(\pi, 1)$. Thus the hypothesis of Lemma 2.1 is satisfied and so $\{D_i(\pi, 1): i \in I\}$ is convergent.

Both the above theorems were used (implicitly) in the construction in [3].

2.5. Theorem. *If condition* (b) *in Theorem* 2.4 *is replaced by*

(b′) *If* $j > i$ *implies* $D_j(\pi_j, 0) = D_i(\pi_j, 0)$

then again $\{D_i : i \in I\}$ *is convergent.*

Proof. By virtue of Theorem 2.4 it suffices to show that $\{D_i(\pi, 0)\colon i \in I\}$ is convergent. If $u \not\equiv w(\text{mod } \pi)$ then $(u, w) \notin A_j(\pi, 0)$ for each j. On the other hand, if $u \equiv w(\text{mod } \pi)$ then fix i and let $j > i$. From condition (a) we have that $(u, w) \in A_j(\pi_j, 0)$ if and only if $(u, w) \in A_i(\pi_j, 0)$ and so $(u, w) \in A_j(\pi, 0)$ if and only if $(u, w) \in A_i(\pi, 0)$. Thus the hypothesis of Lemma 2.1 is fulfilled and we can conclude that $\{D_i(\pi, 0)\colon i \in I\}$ is convergent. $\square$

Remark. When the directed set I is the set ω of integers then it is easy to see that condition (a) of Theorem 2.4 and condition (b′) of Theorem 2.5 can be replaced by the following:

(i) $D_{n+1}(\pi_n, 1) = D_n(\pi_n, 1)$, and

(ii) $D_{n+1}(\pi_{n+1}, 0) = D_n(\pi_{n+1}, 0)$ respectively.

Conditions (i) and (ii) will be needed in Section 3.

3. Disposing of jumps

Recall that a jump in a LOTS $(X, <)$ is a pair of $<$-consecutive elements of X and a gap in X is a pair (A, B) of non-empty subsets of X such that

(1) $A \cup B = X$,

(2) if $x \in A$ and $y \in B$ then $x < y$ and

(3) A has no supremum and B no infimum in X.

If we denote the Dedekind completion of X by DX, then the gaps of X correspond precisely to the elements of $DX - X$. In what follows, when we make a topological statement about the gaps of X we are actually making a statement about the subspace $DX - X$ of DX. We begin this section by obtaining a necessary and sufficient condition for a LOTS with a single jump to be reorderable with a dense order. The proof of Theorem 3.1 is based on an old result from the folklore that a rational open interval is homeomorphic to a rational closed interval.

3.1. Theorem. *Let J_1 be the jumps of a LOTS $(X, <_1)$, $\{a, a^+\} \in J_1$ and suppose $\bigcup J_1 \subset \mathrm{cl}_{DX}(DX - X)$. There exists an order $<_2$ on X which induces the same topology, whose set of jumps $J_2 = J_1 - \{\{a, a^+\}\}$ and in which neither a nor a^+ are extreme elements of X if and only if there exist disjoint non-empty open and closed subsets of $X - \{a, a^+\}$, C_1, C_2, C_3 and C_4 such that*

(1) *$C_1 \cup C_2 \cup \{a\}$ is a neighbourhood of a and $C_3 \cup C_4 \cup \{a^+\}$ is a neighbourhood of a^+.*

(2) $\sup C_1 = \sup C_2 = a$ *and* $\inf C_3 = \inf C_4 = a^+$ *(in the order $<_1$).*

(3) *For each $j \in J$, $j \cap C_i \neq \emptyset$ implies $j \subset C_i$ $(i = 1, 2, 3, 4)$.*

Proof. Suppose sets C_1 and C_2 exist satisfying (1), (2) and (3); it is clear that C_1 and C_2 are contained in the initial $<_1$-interval $(\leftarrow, a)$. The sets C_1 and C_2 are unions of maximal disjoint convex open and closed sets, and it follows from (3) that the suprema and infima of these convex sets are gaps. Let U be an open and closed convex subset of $C_1 \cup C_2 \cup \{a\}$ whose infimum is a gap and whose supremum is a. We define a new order $<'$ on U as follows: If $x_i \in C_i$ then $x_1 <' a <' x_2$; if $x, y \in C_1$ then $x <' y$ if and only if $x < y$; if $x, y \in C_2$, $x < y$ then $x <' y$ if x and y lie in the same member of the family of convex open and closed intervals whose union is C_2 and $y <' x$ otherwise. This clearly defines $<'$ on U. After a similar reordering of some neighbourhood V of a^+ we obtain a new order $<_2$ on X by letting $<_2$ coincide with $<_1$ on $(U \cup V)^c$, with the new orders on U and V and preserving the relative order of U and V in X.

Conditions (1) and (2) ensure that the topology is preserved and (3) implies that no new jumps are created.

Conversely, suppose now that there exists an order $<_2$ on X which induces the same topology on X and whose set of jumps $J_2 = J_1 - \{\{a, a^+\}\}$. We claim that there exists a $<_2$-convex neighbourhood U of a such that if $x \in U$ then $x <_1 a$, for if not, then for each such U we can choose $a_u \in U$ and $a_U >_1 a$. Clearly the net $a_U \to a$ which is a contradiction since $a_U \geqslant_1 a^+$. Now choose $x, y \in U$ so that $x <_2 a <_2 y$ and consider the $<_2$-intervals $(x, a]$ and $[a, y)$. Neither of these sets can be connected otherwise they would be (up to inversion) uniquely orderable (for example, see [2, p. 5]) and hence $a \notin \mathrm{cl}(DX - X)$. Thus both intervals must contain a gap or a jump. If one of them, $(x, a]$ say, contains a jump $\{b, b^+\} \in J_2$ then since $\bigcup J_1 \subset \mathrm{cl}_{DX}(DX - X)$ it follows that each neighbourhood of b and b^+ contains a gap; thus $(x, a]$ also contains a gap. Let s, t be gaps in $(x, a]$ and $[a, y)$ respectively and set $C_1 = (s, a)$ and $C_2 = (a, t)$. Clearly C_1 and C_2 are $<_2$-convex subsets of $X - \{a\}$, and since $a \in \mathrm{cl}\, C_1 \cap \mathrm{cl}\, C_2$ it follows that $<_1 - \sup C_1 = <_1 - \sup C_2 = a$. Furthermore, $C_1 \cup C_2 \cup \{a\}$ is a neighbourhood of a and if there exists some $j \in J_1$ such that $j \cap C_i \neq \emptyset$ and $j \not\subset C_i$ then since C_i is $<_2$-convex it follows that $j \notin J_2$ which is a contradiction. □

3.2. Theorem. *Let J be the set of jumps of a LOTS $(X, <)$, $\{a, a^+\} \in J$ and suppose $\bigcup J \subset \mathrm{cl}_{DX}(DX - X)$. There exist disjoint non-empty open and closed subsets of the initial $<$-interval $(\leftarrow, a)$ satisfying the conditions*

(1) *$C_1 \cup C_2 \cup \{a\}$ is a neighbourhood of a,*

(2) $\sup C_1 = \sup C_2 = a$,

(3) *if $j \in J$ and $j \cap C_i \neq \emptyset$ then $j \subset C_i$ $(i = 1, 2)$,*

if and only if there is a closed (in $DX - X$) set of gaps in $(\leftarrow, a)$ whose only accumulation point in X is a.

Proof. Suppose such C_1 and C_2 exist. Each of them is the union of maximal disjoint open and closed convex sets whose only accumulation point is a; say $C_1 = \bigcup \mathscr{A}_1$ and $C_2 = \bigcup \mathscr{A}_2$. For some regular cardinal λ we can choose $\{A_\alpha : \alpha < \lambda\} \subset \mathscr{A}_1$ such that

(1) $A_\alpha = (l_\alpha, r_\alpha)$ where l_α and r_α are gaps for each $\alpha < \lambda$,

(2) $l_\alpha > r_\beta$ if $\alpha > \beta$,

(3) $\sup\{r_\alpha : \alpha < \lambda\} = a$,

Let $L = \{l_\alpha : \alpha < \lambda\}$. It is easy to see that $\mathrm{cl}_{DX}L - \{a\}$ is the required closed set of gaps whose supremum is a.

Conversely suppose L is a closed (in $DX - X$) set of gaps in $(\leftarrow, a)$ whose only accumulation point in X is a, say $L = \{l_\alpha : \alpha \in I\}$. In L choose a well-ordered increasing closed subset $M = \{l_\alpha : \alpha < \rho\}$, whose supremum is a.

Let $A_\alpha = \{(l_\alpha, l_{\alpha+1}) : \alpha < \rho$ and $\alpha = \beta + 2n$ for some $n \in \omega$ and limit ordinal $\beta\}$.

Let $C_1 = \bigcup\{A_\alpha : \alpha < \rho\}$ and $C_2 = (\leftarrow, a) - C_1$. It is easy to check that C_1 and C_2 satisfy conditions (1), (2) and (3). □

Corollary. *Let J_1 be the set of jumps of a LOTS $(X, <_1)$, $\{a, a^+\} \in J_1$ and suppose $\bigcup J_1 \subset \mathrm{cl}_{DX}(DX - X)$. There exists an order $<_2$ on X inducing the same topology and whose set of jumps $J_2 = J_1 - \{\{a, a^+\}\}$, and in which neither a nor a^+ is an extreme point of $(X, <_2)$ if and only if there exist closed (in $DX - X$) sets of gaps in $(\leftarrow, a)$ and in $(a^+, \rightarrow)$ whose only accumulation points in X are respectively a and a^+.*

Let us say that a LOTS $(X, <)$ has Property (*) if each of its jumps satisfies the hypothesis of Corollary 3.2. The proof of the following theorem is now obvious.

3.3. Theorem. *Let $(X, <)$ be a LOTS with Property (*) and whose set of jumps is discrete, then there exists a dense order on X inducing the same topology (and hence X has a connected ordered compactification).*

However, not every LOTS with Property (*) can be reordered with a dense order, as an example, consider $(R - Q) \times \{0, 1\}$ with the lexicographic order. This space is separable but has continuum-many jumps and hence its weight is $\exp \aleph_0$. Thus any compactification of X would also be separable and of weight $\geqslant \exp \aleph_0$. Again using the relation $w = d + j$ in a LOTS, it follows that any ordered compactification of X must have at least $\exp \aleph_0$ jumps. We now turn to the case of a LOTS with a countable number of jumps. We have seen in Corollary 3.2 that Property (*) is a necessary condition for there to be a dense order on X which induces the same topology. We impose one further restraint in order to obtain a sufficient condition.

Let $(X, <)$ be a LOTS. We define an equivalence relation $\simeq$ on X as follows: $x \simeq x$ and if $x < y$ then $x \simeq y$ if and only if there is no element a of a jump of X such that $x \leqslant a < a^+ \leqslant y$.

If E is a $\simeq$ equivalence class then it is clear that E is convex and $\sup E$ and $\inf E$ (taken in DX) are of one of the following types:

A: a gap of X,

B: an element of a jump of X,

C: an element of X which does not belong to any jump.

We can now prove the main theorem of this paper.

3.4. Theorem. *Let $(X, <)$ be a LOTS with Property $(*)$ and whose set of jumps J is countable. If the set of non-trivial equivalence classes with an extreme point of type C is countable then there is a dense order which induces the topology of X.*

Proof. We enumerate the set of jumps $J = \{j_n : n \in \omega\}$, where $j_n = \{a_n, a_n^+\}$ and the set of equivalence classes with at least one extreme point of type B or C, $\mathscr{C} = \{E_n : n \in \omega\}$. We will "fill in" the jumps one by one using the method of Theorem 3.1 ensuring that a dense limiting order exists which induces the same topology on X.

Step 1: Using the method of Theorem 3.1 we "fill in" in the jump j_1 by choosing disjoint non-empty open and closed subsets C_1^1, C_2^1, C_3^1 and C_4^1 of $X = j_1$ and defining a new order $<_1$ on X as before. Each set C_k^1 $(k = 1, \ldots, 4)$ is the union of maximal disjoint open and closed $<$-convex subsets of X, say $C_k^1 = \bigcup \mathscr{A}_k^1$, whose only accumulation point in X is a_1 or a_1^+. As in Theorem 3.1 we can assume that $C_1^1 \cup C_2^1$ and $C_3^1 \cup C_4^1$ are $<$-convex sets and we further restrict the selection of the sets C_k^1 $(k = 1, \ldots, 4)$ as follows:

We require $C_k^1 \cap E_1 = \emptyset$ $(k = 1, \ldots, 4)$ unless a_1 (respectively a_1^+) is an extreme point of E_1 in which case we choose $C_1^1 \cup C_2^1 \subset E_1$ (respectively $C_3^1 \cup C_4^1 \subset E_1$); we further require that if E is a (non-trivial) equivalence class not of type B or C (i.e. a class both of whose extrema are gaps) and $E \cap C_k^1 \neq \emptyset$ for some k, then for that value of k, $E \subset C_k^1$. It is not hard to see that C_k^1 $(k = 1, \ldots, 4)$ can be chosen to satisfy these restrictions.

Step n: Having chosen (with the restrictions detailed below) the open and closed sets C_k^i for $k = 1, \ldots, 4$ and $i = 1, \ldots, n-1$ and having "filled in" the jumps $j_1, \ldots, j_{n-1}$ by defining new orders $<_1, \ldots, <_{n-1}$ on X as in Theorem 3.1, we proceed to "fill in" the jump j_n as follows. Choose non-empty disjoint open and closed subsets C_1^n, C_2^n, C_3^n and C_4^n of $X - j_n$ and define the order $<_n$ as a modification of the order $<_{n-1}$ as in Theorem 3.1. We can assume that $C_1^n \cup C_2^n$ and $C_3^n \cup C_4^n$ are $<_{n-1}$-convex and we impose the following further conditions on the choice of C_k^n.

(1) We require that $C_k^n \cap \bigcup_{m=1}^n E_m = \emptyset$ unless a_n (respectively a_n^+) is an extreme point of E_m for some $m \in \{1, \ldots, n\}$ in which case for this value (these two values) of m we choose $C_1^n \cup C_2^n \subset E_m$ (respectively $C_3^n \cup C_4^n \subset E_m$).

(2) If E is a non-trivial equivalence class both of whose extreme points are gaps and $E \cap C_k^n \neq \emptyset$ for some $k \in \{1, \ldots, 4\}$ then for this value of k we require that $E \subset C_k^n$.

(3) If $j_n \cap \bigcup_{i=1}^{n-1} \bigcup_{k=1}^4 C_k^i = \emptyset$ then there exists a $<_{n-1}$-convex neighbourhood U_n of a_n and a_n^+ such that $U_n \cap \bigcup_{i=1}^{n-1} \bigcup_{k=1}^4 C_k^i = \emptyset$ and we require that

$$\inf U_n <_{n-1} \inf(C_1^n \cup C_2^n) <_{n-1} \sup(C_3^n \cup C_4^n) <_{n-1} \sup U_n.$$

If $j_n \cap \bigcup_{i=1}^{n-1} \bigcup_{k=1}^4 C_k^i \neq \emptyset$ then let i_0 be the maximal value of i $(\leq n-1)$ such that $j_n \cap C_k^i \neq \emptyset$ (for some $k = 1, \ldots, 4$) and let k_0 be the unique value of k such that $j_n \cap C_{k_0}^{i_0} \neq \emptyset$ and hence such that $j_n \subset C_{k_0}^{i_0}$. Furthermore, since $C_{k_0}^{i_0}$ is the union of a family of maximal disjoint open and closed $<_{i_0-1}$-convex sets, $C^{i_0}_{k_0} = \bigcup \mathscr{A}^{i_0}_{k_0}$ say, it follows that j_n is contained in some unique element $A^{i_0}_{k_0} \in \mathscr{A}^{i_0}_{k_0}$. Let U_n be a $<_{n-1}$-convex neighbourhood of a_n and a_n^+ such that $U_n \subset A^{i_0}_{k_0}$ and $U_n \cap \bigcup_{i=i_0+1}^{n-1} \bigcup_{k=1}^{4} C^i_k = \emptyset$. We require that

$$\inf U_n <_{n-1} \inf(C^n_1 \cup C^n_2) <_{n-1} \sup(C^n_3 \cup C^n_4) <_{n-1} \sup U_n$$

We have thus defined orders $<_m$ on X for each $m \in \omega$. It is clear that the construction satisfies conditions (i) and (ii) of the Remark following Theorem 2.5, and so a limiting order $<_\omega$ exists. We show that

(a) $<_\omega$ is a dense order,

(b) $<_\omega$ induces the same topology on X as $<$.

(a) Suppose that $<_\omega$ is not a dense ordering of X, then there exists a jump $\{x, x^+\}$ in $(X, <_\omega)$. From the definition of $<_\omega$ we see that there is some $n_0 \in \omega$ such that if $n \geqslant n_0$ then $x <_n x^+$. Furthermore, $\{x, x^+\} \notin J$ since it is clear that all the jumps of $(X, <)$ are "filled in". Let z be such that $x <_{n_0} z <_{n_0} x^+$. At some future step $m > n_0$ we must have $z \in C^m_k$ for some $k = 1, \ldots, 4$ (otherwise z would stay between x and x^+, which is a contradiction) and z is not $<_n$-between x and x^+ for each $n \geqslant m$. Since C^m_k is the union of a family of maximal disjoint $<_{m-1}$-convex open and closed sets, $C^m_k = \bigcup \mathscr{A}^m_k$ say, it follows that z lies in some unique element $A^m_k \in \mathscr{A}^m_k$. It is then not hard to see, since $x <_m x^+$ but z is not $<_m$-between x and x^+, that we must have $A^m_k \subset (x, x^+)_{m-1}$ (the open $<_{m-1}$-interval from x to x^+); hence it follows that $a_m >_{m-1} x^+$ (or alternatively $a^+_m <_{m-1} x$) and that A^m_k lies $<_m$-between a_m and a^+_m. We assume that $a_m >_{m-1} x^+$ and then the situation may be illustrated as in Fig. 1. Since $C^m_1 \cup C^m_2$ is $<_{m-1}$-convex it follows that $x^+ \in C^m_1 \cup C^m_2$. If $x^+ >_m a_m$ (i.e. if $x^+ \in C^m_2$ the set which is used to "fill in" the jump j_m) then a_m will remain between x and x^+ in all further steps, which is a contradiction. Thus $x^+ \in C^m_1 = \bigcup \mathscr{A}^m_1$, say $x^+ \in A^m_1 \in \mathscr{A}^m_1$. Now let y be any element of $A^m_1 \cap (x, x^+)_m$ (Fig. 2).

A repetition of the above process must now be employed in order, at some future step $r > m$, to remove y from the interval $(x, x^+)_{r-1}$. This may involve moving y into the jump j_r to the right of x^+ or moving x^+ into the jump j_r to the left of y,

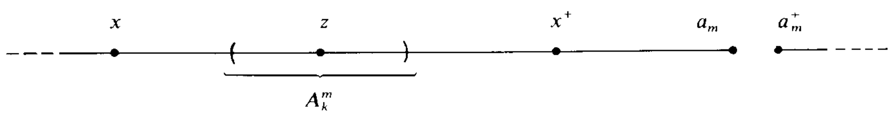

Fig. 1. Order is $<_{m-1}$; A^m_k is one of the sets used to "fill in" jump j_m.

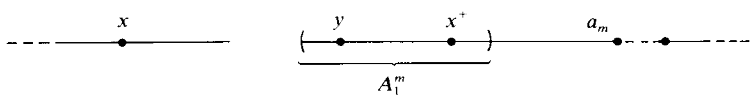

Fig. 2. Order is $<_m$.

but in either case $j_r \subset A_1^m$, and condition (3) applies. Let $B = \bigcup_{k=1}^4 C_k^r \cup j_r$. Then $\inf B >_{r-1} \inf A_1^m$ and it is clear that if $w \in A_1^m$ and $w <_{r-1} \inf B$, then w remains between x and x^+ in all succeeding steps (Fig. 3).

(b) Let $x \in X$. It is clear from the construction that the topology induced on X by each of the orders $<_n$ is identical to the original topology. Thus if x has a $<$-neighbourhood U which intersects at most a finite number of $V_n = \bigcup_{k=1}^4 C_k^n \cup j_n$, it follows that after some finite number, say m_0, of steps, the order on U remains unchanged and so $<_n = <_\omega$ on U if $n \geq m_0$. Hence the $<_\omega$-order topology and the $<$-order topology coincide on U.

Now suppose that every $<$-neighbourhood of x intersects an infinite collection of V_n. We will show that there is a $<$-neighbourhood base at x of $<$-convex sets which are also $<_\omega$-convex, $<_\omega$-neighbourhoods of x. There are two cases to consider:

(i) $x \in \bigcap_{m \in I} V_m$ for some infinite set $I \subset \omega$. We note first that if E is an equivalence class both of whose extrema are gaps and $x \in E$, then E is a $<$-neighbourhood of x and the orders $<$ and $<_\omega$ coincide on E because of condition (2) above. If x is not an element of such an equivalence class then we claim that $\{x\} = \bigcap_{m \in I} A^m$ where A^m is the unique element of $\bigcup_{k=1}^4 \mathscr{A}_k^m$ to which x belongs for each $m \in I$. If not, then there exists $y \neq x$ such that $y \in A^m$ for each $m \in I$. If there is no jump $<$-between x and y then for some n_0, x, $y \in E_{n_0}$. But then it follows from condition (1) of the construction that for all but at most one $n \geq n_0$, x, $y \notin V_n$ which is a contradiction. Thus there must be at least one jump, j_n say, $<$-between x and y. Since A^m is an $<_{m-1}$-interval it follows that $j_n \in A^m$ for each $m = 1, \ldots, n-1$, but at step n it is clearly impossible for both x and y to lie in A^n. Thus $\{x\} = \bigcap_{m \in I} A^m$. It is clear from the construction that each A^m is convex in each of the orders $<_n$ and so it follows from Theorem 2.3 that A^m is $<_\omega$-convex for each m. Furthermore, since $\{A^m : m \in I\}$ is a nested family, in order to show that it is a $<_\omega$-local base at x, we need only prove that each of the A_m is a $<_\omega$-neighbourhood of x; however, this is easily seen to follow from condition (3).

(ii) Now suppose that x is an element of no more than a finite number of V_n. It is clear that without loss of generality we can assume that $x \notin V_n$ for all $n \in \omega$, but x is an accumulation point of an infinite family $\{V_n : n \in F\}$, with say $x < \inf V_n$, for all $n \in F$. Let $\{V_{k_n} : k_n \in F' \subset F\}$ be a disjoint subfamily with $<$-infimum x. Then $\{[x, <\text{-}\inf V_{k_n}) : n \in \omega\}$ is a family of $<$-convex sets which are also $<_\omega$-convex and each of whose $<_\omega$-infimum is x. This family may either be combined with a similar one to the left of x to give a (nested) local base of $<$- and $<_\omega$-convex sets at x, or

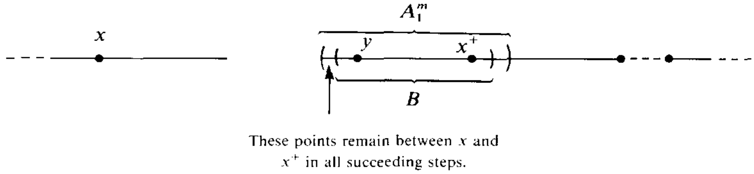

Fig. 3. Order is $<_{r-1}$.

with a nested family of intervals $\{(x_\alpha, x]: \alpha < \gamma\}$ each contained in some non-trivial equivalence class to give a local base of $<$-convex and $<_\omega$-convex sets directed by $\gamma \times \omega$. $\square$

We note that if X is a first countable LOTS in which the gaps are dense in the jumps ($\bigcup J \subset \mathrm{cl}_{DX}(DX - X)$) then X will have Property (*). Thus we have the following corollaries (the second being well known).

3.5. Corollary. *Let $(X, <)$ be a first countable LOTS with a countable set of jumps, whose set of equivalent classes is countable and in which the gaps are dense. There is a dense order which induces the topology of X.*

3.6. Corollary. *A second countable ordered space with a dense set of gaps is homeomorphic to a dense subset of the real line.*

Theorems 3.3 and 3.4 can be combined to give the following result which is inspired by the generalization in [4] of the principal theorem of [3]:

3.7. Theorem. *Let $(X, <)$ be a LOTS with Property (*) and whose set of jumps J is σ-discrete. If the set of non-trivial equivalence classes with an extreme point of Type C is also σ-discrete then there is a dense order which induces the topology of X.*

Proof. Let $\mathscr{C}$ be the set of non-trivial equivalence classes with an extreme point of Type C. It is not hard to verify that both J and $\mathscr{C}$ can be expressed as countable unions of nested collections of discrete families, say

$$J = \bigcup \{J_n : n \in \omega\} \quad \text{where } J_n \subset J_{n+1} \text{ for each } n$$

and

$$\mathscr{C} = \bigcup \{\mathscr{C}_n : n \in \omega\} \quad \text{where } \mathscr{C}_n \subset \mathscr{C}_{n+1} \text{ for each } n$$

The proof now follows that of Theorem 3.4, at step n we use Theorem 3.3 to "fill in" all the jumps of the discrete family $J_n - \bigcup \{J_i : i = 1, \ldots, n-1\}$. We omit the laborious but by now familiar details. $\square$

We recall that a metric space has a σ-discrete base and use this fact to prove the following:

3.8. Lemma. *If X is a metric linearly ordered topological space then both the set of jumps and the set of non-trivial $\simeq$ equivalence classes are σ-discrete.*

Proof. We consider first the set of jumps $J = \{j_\alpha : \alpha < \lambda\}$ say. Let $\mathscr{B} = \bigcup_{n \in \omega} \mathscr{B}_n$ be a σ-discrete base for the space X. For each $j_\alpha \in J$, $j_\alpha = \{a_\alpha, a_\alpha^+\}$ there exist U_α, $V_\alpha \in \mathscr{B}$ such that

$$a_\alpha \in U_\alpha \subset (\leftarrow, a_\alpha] \quad \text{and} \quad a_\alpha^+ \in V_\alpha \subset [a_\alpha^+, \rightarrow)$$

Let $\mathcal{U} = \{U_\alpha, V_\alpha : \alpha \in \lambda\} \subset \mathcal{B}$. Clearly $\mathcal{U}$ is σ-discrete, say $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ where $\mathcal{U}_n$ is a discrete family for each n. Define $J_{n,m} = \{j_\alpha \in J: U_\alpha \in \mathcal{U}_n, V_\alpha \in \mathcal{U}_m\}$, then it is clear that $J = \bigcup_{n,m} J_{n,m}$ and each $J_{n,m}$ is discrete since $\mathcal{U}_n$ and $\mathcal{U}_m$ are discrete.

Now let $\mathcal{A}$ be the set of non-trivial $\simeq$ equivalence classes. $\mathcal{A}$ is a family of intervals with non-empty interior and since each two elements of $\mathcal{A}$ have at least one jump between them, it follows that no point of X is in the closure of two elements of $\mathcal{A}$. Suppose $\mathcal{A} = \{A_\alpha : \alpha < \eta\}$, then for each $\alpha < \eta$ we can choose $U_\alpha \in \mathcal{B}$ such that $U_\alpha \subset A_\alpha$. Let $\mathcal{U} = \{U_\alpha : \alpha < \eta\} \subset \mathcal{B}$. $\mathcal{U}$ is clearly σ-discrete,

$$\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n \quad \text{say.}$$

Let $\mathcal{A}_n = \{A_\alpha \in \mathcal{A}: U_\alpha \in \mathcal{U}_n\}$

Clearly $\mathcal{A} = \bigcup_{n\in\omega} A_n$ and it is not hard to see that if A_n were not discrete for some n, then $\mathcal{U}_n$ would not be discrete either. $\square$

3.9. Corollary. *Every metric LOTS with Property (*) has a dense order which induces the topology.*

The proof of the following theorem is now straightforward and will be left to the reader.

3.10. Theorem. *Let $(X, <)$ be a metrizable LOTS whose set of jumps is J and whose set of extreme points is C. X is densely orderable and hence has a connected ordered compactification if and only if*

(1) *X has no proper subsets which are compact and open, and*

(2) *There are at most two points of $C \cup (\bigcup J)$ which are not in the closure of $DX - X$.*

It is easy to see that conditions (1) and (2) in Theorem 3.10 are equivalent to the following:

There are at most two points of $C \cup (\bigcup J)$ which are not in the closure of $DX - X$ and furthermore if two such points exist then they are not in the same component of X if X is disconnected.

It was shown in [1] that every strongly zero-dimensional metric space is orderable. Hence we have the following.

3.11. Theorem. *If X is a strongly zero-dimensional metric space with no compact open subsets then X is densely orderable and hence has a connected compactification.*

Property (*) and the inequality $j \leqslant d$ are necessary conditions for a LOTS to be densely orderable. While we believe that they are probably not sufficient conditions, we know of no counterexample.

We note that metric LOTS which do not have Property (*) may have connected compactifications without having connected orderable compactifications; $[0, 1) \cup (1, 2] \cup [3, 4)$ with the relative topology of the reals is such a space.

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